Math 10B with Professor Stankova
Worksheet, Discussion #8; Tuesday, 2/13/2018
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1 Conditional Probability

1.1 Examples

1. I flip a fair coin 12 times. What is the probability that exactly 10 heads appear given that at least two heads appeared?

> **Solution:** We need to find the probability $P(10H| \geq 2H)$ and by definition, we need to find $P(10H\cap \geq 2H)$ and $P(\geq 2H)$. The event that we get at least 2 heads at exactly 10 heads is just 10 heads and we can calculate the probability that we have at least 2 heads by complementary counting. This probability is $1 - P(< 2H) = 1 - P(0H) - P(1H) = 1 - \frac{1}{2^12} - \frac{12}{2^12} = 1 - \frac{13}{2^{12}}$. Thus, we have that
>
> $$P(10H| \geq 2H) = \frac{P(10H)}{P(\geq 2H)} = \frac{\frac{\binom{12}{10}}{2^12}}{1 - \frac{13}{2^{12}}} = \frac{\binom{12}{10}}{2^12 - 13}.$$

2. Out of those brought to court, there are 60% which are actually guilty. Of those that are guilty, 95% of them are convicted. But there are 1% of innocent people who get falsely convicted. What is the probability that you are actually innocent given that you are convicted?

> **Solution:** We use Bayes rule which tells us that $P(Innocent|Convicted)$
>
> $$= \frac{P(Convicted|Innocent)P(Innocent)}{P(Convicted|Innocent)P(Innocent) + P(Convicted|Guilty)P(Guilty)}$$
>
> $$= \frac{0.01 \cdot 0.4}{0.01 \cdot 0.4 + 0.95 \cdot 0.6} \approx 0.7\%$$

1.2 Problems

3. **TRUE** False Bayes Theorem can be proven through the definition of conditional probability.

4. Out of 330 male students and 270 female students in 10B, 210 of the men and 180 of the women took 10A with Zvezda last semester. What is the probability that a randomly person is a female given that they took 10A with Zvezda?

Solution: We have that $P(F|Zvezda) = \frac{P(F \cap Zvezda)}{P(Zvezda)} = \frac{P(F \cap Zvezda)}{P(Zvezda \cap M) + P(Zvezda \cap F)} = \frac{180}{210+180} = \frac{18}{39} = \frac{6}{13}$.

5. You ask your neighbor to water your plant. Without water, there is a 90% chance the plant with die. With water, this percentage drops to 30%. You are 80% sure that your neighbor will remember to water your plant. What is the probability that your neighbor forgot given that it died?

Solution: We want to find $P(Forgot|Dead) = \frac{P(Forgot \cap Dead)}{P(Dead)}$

$$= \frac{P(Dead|Forgot)P(Forgot)}{P(Dead|Forgot)P(Forgot) + P(Dead|NotForgot)P(NotForgot)}$$

$$= \frac{0.9 \cdot 0.2}{0.9 \cdot 0.2 + 0.3 \cdot 0.8} = \frac{18}{42} = \frac{3}{7}.$$

6. I have two boxes of apples and oranges. In box 1, there are 5 oranges and 6 apples, in box 2 there are 6 oranges and 5 apples. I randomly pick a box and then in this box randomly pick a fruit. What is the probability that I picked box 1 given that I picked an orange?

Solution: We have that

$$P(Box1|Orange) = \frac{P(Box1 \cap Orange)}{P(Orange)}$$

$$= \frac{P(Orange|Box1)P(Box1)}{P(Orange|Box1)P(Box1) + P(Orange|Box2)P(Box2)} = \frac{5/11 \cdot 1/2}{5/11 \cdot 1/2 + 6/11 \cdot 1/2} = \frac{5}{11}.$$

7. An exam has a 99% chance of testing positive if you have the disease and 1% chance of testing positive if you do not have the disease. Give that 0.5% of people have this disease, what is the probability that you have the disease given that you tested positive?

Solution:

$$P(Sick|Positive) = \frac{P(Positive|Sick)P(Sick)}{P(Positive|Sick)P(Sick) + P(Positive|Notsick)P(Notsick)}$$

$$= \frac{0.99 \cdot 0.005}{0.99 \cdot 0.005 + 0.01 \cdot 0.995} \approx 0.33.$$

8. About 1 in a million Americans play in the NBA. Suppose that 90% of NBA players are very tall and 2% of all other Americans are very tall. What is the probability someone is in the NBA given that they are very tall?

Solution: We have that

$$P(NBA|Tall) = \frac{P(Tall|NBA)P(NBA)}{P(Tall|NBA)P(NBA) + P(Tall|NotNBA)P(NotNBA)}$$

$$= \frac{0.9 \cdot 1/10^6}{0.9 \cdot 1/10^6 + 0.02 \cdot (1 - 1/10^6)} \approx 0.0045\%.$$

9. About 2/3 of drivers use their cell phone while driving. Suppose that you are 5 times more likely to get into an accident if you text and drive, and if you don't use your cell phone, you have a 1% chance of getting into an accident. What is the probability that someone was texting given that they got into an accident?

Solution: We have that

$$P(Text|Accident) = \frac{P(Accident|Text)P(Text)}{P(Accident|Text)P(Text) + P(Accident|NoText)P(NoText)}$$

$$= \frac{0.05 \cdot 2/3}{0.05 \cdot 2/3 + 0.01 \cdot 1/3} = \frac{10}{11}.$$

True/False

10. **TRUE** False Among the problems we considered in class, a multi-stage process can be encoded (and solved) by either dependent choice or independent choice at each stage, or split into mutually exclusive cases.

11. True **FALSE** We can turn any counting problem into a problem using the product rule or the sum rule.

12. True **FALSE** Reversing the order of stages in a process does not affect the difficulty or efficiency of solving the problem.

13. True **FALSE** $A \times B \times C$ for some sets A, B, C is another set made of all possible triplets (x, y, z) where x, y, z are any elements of the three sets.

Solution: We need $x \in A, y \in B, z \in C$.

14. **TRUE** False We solved in class the problem of finding the size of the power of a set by setting up a multi-stage process with 2 independent choices at each stage.

Solution: Each stage was deciding whether or not to put an element in the subset or not.

15. **TRUE** False The product rule for counting usually applies if we use the word "AND" between the stages of the process, while the sum rule for counting is usually used when we can finish the whole process in different ways/algorithms and we use the word "OR" to move from one way to another.

16. True **FALSE** Among the problems we considered so far in class, a multi-stage process can be encoded (and solved) by either dependent choice or independent choice at each stage, split into mutually exclusive cases, or split into "good" and "bad" cases.

Solution: We can also use PIE.

17. **TRUE** False Counting problems where the phrase "at least once" appears may indicate using the complement, or equivalently, counting all cases and subtracting from them all "bad" cases.

18. **TRUE** False Tree diagrams present a visual explanation of a situation, but unless one draw the full tree diagram to take into account all possible cases, the problem is not solved and will need more explanation/justification.

19. True **FALSE** To find how many natural numbers $\leq n$ are divisible by d, we calculate the fraction n/d and round up in order to not miss any numbers.

Solution: We round the fraction down, not up.

20. **TRUE** False We use 1 more than the ceiling (and not the floor) function in the statement of the Most General PHP because, roughly, we want to have one more pigeon that the ratio of pigeons to holes in order to "populate" a hole with the desired number of pigeons.

21. True **FALSE** It is always true that $\lfloor x \rfloor \le x \le \lceil x \rceil$ for any real number x, but equality of the two extreme terms of this inequality is never possible.

Solution: There is equality when x is an integer.

22. **TRUE** False Proof by contradiction can be used to justify any version of the PHP.

23. **TRUE** False A phrase of the type "at least these many objects" indicates what the pigeons should be in a solution with PHP, while "share this type of property" points to what the holes should be and how to decide to put a pigeon into a hole.

Solution: The objects in "at least these many objects" should be the pigeons and properties in "share this type of property" should be the holes.

24. **TRUE** False Erdos-Szekeres Theorem on monotone sequences is a generalization of the class problem on existence of an increasing or a decreasing subsequence of a certain length, and its proof assumes that one of two possibilities is not happening and shows that the other possibility must then occur.

Solution: This is how we proved that in a sequence of 10 people, there must be at least 4 in an increasing or decreasing order.

25. True **FALSE** Any version of the PHP implies existence of certain objects with certain properties and shows us how to find them.

Solution: It doesn't tell us how to find them, only that it exists.

26. **TRUE** False To prove that there are some two points exactly 1 inch apart colored the same way on a canvas painted in black and white, it suffices to pick an equilateral triangle of side 1 in on this canvas and apply PHP to its vertices being the pigeons and the two colors (black and white) being the holes.

27. True **FALSE** To show that a conclusion does not follow from the given conditions, we need to do more work than just show one counterexample.

Solution: All we need to do is to show one counterexample.

28. **TRUE** False A counterexample is a situation where the hypothesis (conditions) of a statement are satisfied but the conclusion is false.

29. True **FALSE** The k-permutations of an n-element set are a special case of the k-combinations of this set.

30. **TRUE** False An identity is an equality that is always true for any allowable values of the variables appearing in the equality.

31. **TRUE** False An ordered k-tuple can be thought of some permutation of k elements, while an unordered k-tuple can be thought of a combination of k elements (perhaps, coming from a larger set).

32. **TRUE** False To prove some identity combinatorially roughly means to count the same quantity in two different ways and to equate the resulting expressions (or numbers).

33. **TRUE** False One good reason for 0! to be defined as 1 is for the general formula with factorials for C(n,k) to also work for k=0.

34. True **FALSE** The number of combinations C(n,k) is the number of permutations P(n,k) divided by the number of permutations P(n,n).

Solution: It should be divided by P(k, k).

35. True **FALSE** The symmetry of permutations can be seen in the identity P(n,k)=P(n,n-k) for all integer $n, k \geq 0$.

Solution: There is a symmetry of combinations, not permutations.

36. True **FALSE** The number of ways to split 10 people into two 5-person teams to play volleyball is $\frac{10!\cdot 10!}{2}$ because forgetting the 2 in the denominator would result in an overcount by a factor 2, which can be interpreted as an additional assignment of a court to each team on which to play (not required by the problem!).

Solution: The answer is just $\binom{10}{5}$.

37. **TRUE** False It is possible to use Calculus to prove combinatorial identities.

Solution: We can take the derivative of $(1+x)^n$ to get the equality

$$1\binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n} = n2^{n-1}.$$

38. True **FALSE** Interpreting the same quantity in two different ways is not useful in proving binomial identities because, ultimately, one of the interpretations is harder (or impossible!) to calculate on its own.

Solution: Proving binomial identities through two interpretations is often the slickest way to do it.

39. **TRUE** False The binomial coefficients appear in Pascal's triangle, as coefficients in algebraic formulas, and as combinations.

40. True **FALSE** The alternating sum of the numbers in an even-numbered row of Pascal's triangle is zero for the simple reason that Pascal's triangle is symmetric across a vertical line; but the same statement for an odd-numbered row requires some deeper analysis since the numbers there do not readily cancel each other.

Solution: The alternating sum of an odd-numbered row is zero because of symmetry. Note that an odd numbered row has an even number of elements, for instance row 5 has the elements $\binom{5}{0}, \binom{5}{1}, \binom{5}{2}, \binom{5}{3}, \binom{5}{4}, \binom{5}{5}$.

41. **TRUE** False The basic combinatorial relation satisfied by binomial coefficients that makes it possible to identify all numbers in Pascal's triangle as some binomial coefficients can be written as $\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$ for $n, k \geq 1$.

42. **TRUE** False The formula $1+2+3+\cdots+n = \binom{n+1}{2}$ for $n \geq 1$ is a special case of the Hockeystick Identity $\binom{k}{k}+\binom{k+1}{k}+\binom{k+2}{k}+\cdots+\binom{n}{k} = \binom{n+1}{k+1}$ for $n \geq k \geq 0$.

> **Solution:** Take $k = 1$ and then we get
> $$\binom{1}{1} + \binom{2}{1} + \binom{3}{1} + \cdots + \binom{n}{1} = 1 + 2 + \cdots + n = \binom{n+1}{1+1} = \binom{n+1}{2}.$$

43. **TRUE** False The binomial coefficients first increase from left to right along a row in Pascal's triangle, but then they decrease from the middle to the end of the row.

44. True **FALSE** $k\binom{n}{k} = n\binom{n-1}{k-1}$ unless $k > n$.

> **Solution:** This is true for all n, k, namely even when $k > n$. When $k > n$, then both sides are 0. For $k \leq n$, this is an application of Question 22 via
> $$\binom{n}{k}\binom{k}{1} = \binom{n}{1}\binom{n-1}{k-1}.$$
> Another way to think about this is creating a team of size k with a captain. First we can select the team and then a captain, or we can first select the captain and then select the rest of the team.

45. **TRUE** False The coefficient of x^3y^2 in $(x+y)^6$ is 0 because $2+3 \neq 6$; yet, it appears twice in the expanded form of $(x+y)^5$.

> **Solution:** The coefficient of x^3y^2 in $(x+y)^6$ is 0 since $2+3 \neq 6$. But, it appears two times in $(x+y)^5$, namely as $\binom{5}{2}x^2y^3$ and $\binom{5}{3}x^3y^2$.

46. **TRUE** False We can use the Binomial Theorem to prove all sorts of binomial identities, provided we recognize what x, y, and n to plug into it.

> **Solution:** We used the Binomial coefficient to prove that the sum of all of the elements in a row of the Pascal's triangle was 2^n by plugging in $x = y = 1$ and we proved that the alternating sum is 0 by plugging in $x = 1, y = -1$.

47. **TRUE** False In general, it is harder to handle balls-into-boxes problems where the function must be surjective than where the function is injective or there are no restrictions on it.

48. **TRUE** False The number of k combinations from n elements with possible repetition is $\binom{n+k-1}{n-1}$ and it matches the answer to the problem of distributing k identical biscuits to n hungry (distinguishable) dogs.

49. **TRUE** False The number of 7-letter English words (meaningful or not, with possible repetition of letters) is not equal to the ways to distribute 7 equal bonuses to 26 people (with possible multiple-bonus winners).

50. **TRUE** False The equation $x_1 + x_2 + x_3 + x_4 = 10$ in natural numbers has as many solutions as trying to feed 4 (different) dogs with 6 (identical) biscuits.

Solution: Since we are working with natural numbers, each of the numbers has to be at least 1 so we can first feed each dog one biscuit and then we have 6 left to distribute.

51. True **FALSE** The expression $(x + y + z + t)^{2018}$ has $\binom{2020}{3}$ terms after multiplying through but before combining similar terms, and 4^{2018} terms after combining similar terms.

Solution: It is 4^{2018} terms before combining similar terms and $\binom{2021}{3}$ terms after combining terms.